# Singular Perturbed Monotone & Competitive Systems

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# Monotone Dynamical Systems (MDS)

- **Brief History**:

  • *Monotone iteration scheme*.
  —- Bieberbach ([1912])
  —- Courant and Hilbert (Methoden der Mathematischen Physik [1930]);

  • *Comparison principle of ODEs and PDEs*.
  —- M. Müller ([1926]), E. Kamke ([1932]);
  —- E. Hopf ([1927])

  • *Monotonicity fully integrated with Dynam. Sys. ideas*.
  —- M. W. Hirsch
  (A remarkable series of works [82-91], survey [BAMS84,Crelle88])

- Establishment of the theory of MDS by Hirsch.

# Ordered Banach Space $(V, V_+)$

- Phase space $(V, V_+)$.
  - A closed convex solid cone $V_+ \subset V$ ($\mathrm{Int} V_+ \neq \emptyset$).
  - A (strong) ordering on $V$ as ($\forall x_1, x_2 \in V$):

$$x_2 \leq x_1 \Longleftrightarrow x_1 - x_2 \in V_+;$$
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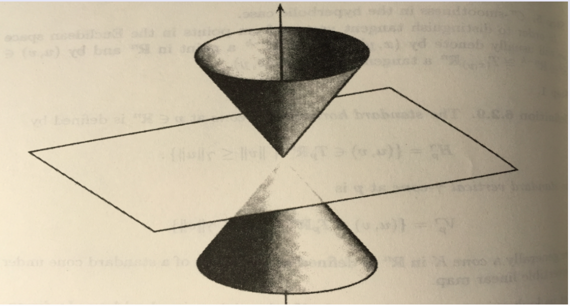

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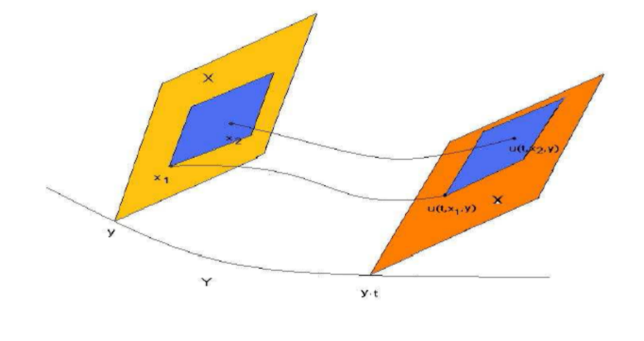
- Simple examples:
  - $(\mathbb{R}^n, \mathbb{R}^n_+), \quad x \leq y \Leftrightarrow x_i \leq y_i, \forall i;$
  - $(C(\bar{\Omega}), C_+(\bar{\Omega})), \quad u \leq v \Leftrightarrow u(x) \leq v(x), \forall x \in \bar{\Omega}.$
  - $(W^{k,p}(\Omega), W^{k,p}_+(\Omega)), \quad u \leq v \Leftrightarrow u(x) \leq v(x), \mathrm{a.e.}\, x \in \Omega.$

# Monotone Dynamical Systems (MDS)

- Monotone dynamical system $\Pi_t$:
  $u(t, x_1) \leq u(t, x_2)$ whenever $t \geq 0$ and $x_1 \leq x_2$.
- Strongly Monotone dynamical system $\Pi_t$:
  - $\Pi_t$ is monotone;
  - $u(t, x_1) \ll u(t, x_2)$ whenever $t > 0$ and $x_1 < x_2$.



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  • Systems of cooperative ODEs;

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  • Systems associated with positive or negative feedback.

# Conclusions of strongly MDS

- Autonomous Cases: (Tendency not to be chaotic)

  • Limit-sets dichotomy (Hirsch[BAMS84,Crelle88])
  —- ($x \leq y \Longrightarrow$ either $\omega(x) \ll \omega(y)$ or $\omega(x) = \omega(y) \subset E$)

  • Generic convergence to the set of equilibria.
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  • For smooth systems: Generic convergence to equilibria
  —- (Poláčik [89], Smith&Thieme [92]).

  • Complicated dynamics restricted to Cod-1 invariant mfds (at most countably many such mfds, with sort of instability).
  —— (Hirsch[86], Dancer and Poláčik [MAMS98]);

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- Autonomous/Periodic Cases:
  Global dynamics with various structures:
  —— Brunovsky, Dancer, Hess, Matano, Smith, Mallet-Paret, Jiang, Mierczynski, Sell, Smillie, Fiedler, Sontag, Wu,...

## Conclusions of strongly MDS

- Skew-product semiflows: (Multiple External Frequency)
  - No generic convergence;
  - The dynamics on $\omega$-limit sets can be very complicated;
  - Almost automorphic phenomenon.

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- Dynamics under amenable structures:

  —— Orbital Stability (Alikakos, Bates, Jiang, Zhao ...)

  —— Sublinearity (Obaya et al. [05,10,12,14]);

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- Group Actions on Skew-product semiflows:

  —— Symmetry& Monotonicity w.r.t. connected Group (W.[09], Cao, Gyllenberg, W. [JEMS16]);
  —— Phase-translation Group (Liu and W.[TAMS12]).

# Competitive Dynamical Systems (CDS)

- Competition. —- An increase of any one species does not tend to increase the per capita growth rate of any other species.
- CDS —- A dynamical system that describes the phenomena of competition.

# Competitive Dynamical Systems (CDS)

- Examples:

  - Systems of Kolmogorov ODEs: (Say, LV-systems)

$$\dot{x}_i = x_i f_i(x), \qquad 1 \le i \le n;$$
$$x = (x_1, x_2, \ldots, x_n) \in C := \{x_i \ge 0, \;\; 1 \le i \le n\}.$$

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  - Discrete-time Mappings. $T : C \to C$ continuous s.t.

$$T_i(x) = x_i G_i(x), \quad x \in C, i = 1, \cdots, n.$$

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$$\partial_t u_i = \triangle u_i + u_i f_i(t, x, u), \qquad u \ge 0, 1 \le i \le n,$$

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  with boundary conditions.

- "Competition" is modeled by the assumption

$$\frac{\partial f_i}{\partial x_j} \le 0 \quad \text{or} \quad \frac{\partial G_i}{\partial x_j} \le 0.$$

# Conclusions of CDS

- Competitive systems are "dissipative systems".
—- $\exists$ a "global" compact attractor, say $\Sigma$, attracting all nontrivial orbits.
- Chaotic behavior observed (Smale [76], Xiao [10]).
- CoD-1 topological structure of $\Sigma$ – Carrying Simplex:

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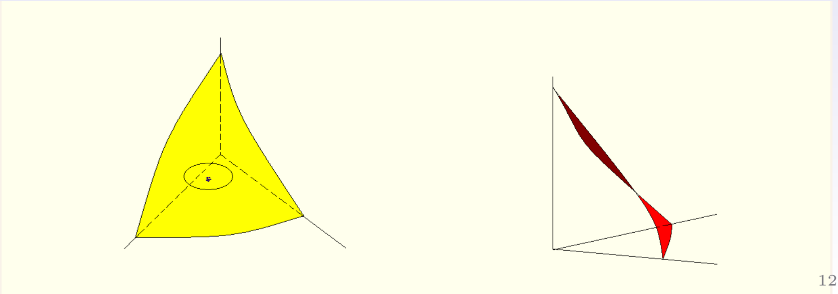
Fig: "front" and "side" views of $\Sigma$ for a 3-D competitive LV system with an attracting periodic orbit



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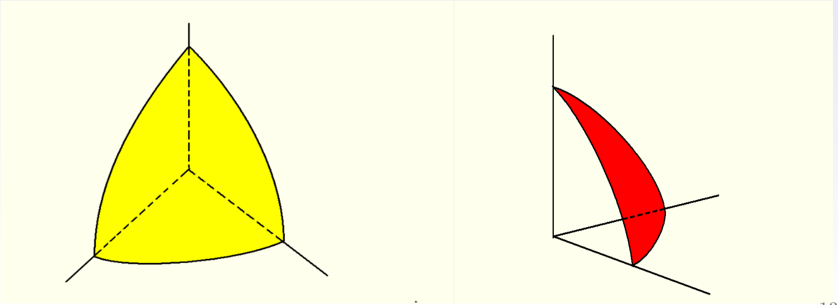

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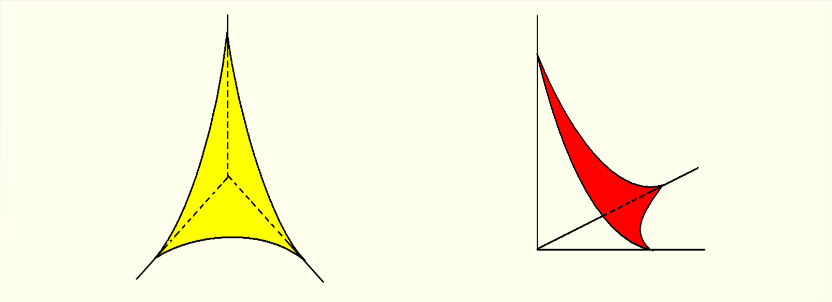

# Simple Examples

- Logistic differential equations:

$$dx/dt = rx(\sigma - x), \quad r, \sigma > 0.$$

  - $\Gamma = [0, \sigma]$ and $\Sigma = \{\sigma\}$.
  - $\Sigma$ is located at the carrying capacity $\sigma$.

- Discrete Ricker-type competition:

$$T : [0, +\infty) \to [0, +\infty); Tx \mapsto xe^{b-ax}, \quad a, b > 0.$$

  - If $b \leq 2$, $\Gamma = [0, \frac{b}{a}]$ and $\Sigma = \{\frac{b}{a}\}$;
  - If $b > 2$, no carrying simplex;
  - If $b \gg 2$, Chaotic behavior occurs.

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  - 3D-competitive sys: Poincaré-Bendixson Type Thm.
  - 3D-LV sys.: Limit & heteroclinic cycles on $\Sigma$.

  —— M. Zeemann, D. Xiao, May-Leonard, Sigmund, Hofbauer&So, Gyllenberg, M. Han, P. Yu, Lu&Luo, Yan, W. ...

# Conclusions of CDS

- Discrete-time CDS:

▲ Existence of $\Sigma$:

• Diffeomorphism: (conjectured by Smith[86])

—— Time-periodic equations (Jiang & W. [JDE02]);

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- Non-autonomous & Random CDS:

—— (Shen & W.[JDE08], Fang, Gyllenberg & W. [SIMA13])

# Difference/Relation between MDS & CDS

- Difference/Relation between MDS & CDS:
  - ▲ Expected dynamics:
    - MDS: Generic convergence to equilibria/cycles
    - CDS: Chaotic behavior observed
  - ▲ Order-preserving phenomenon:
    - MDS: Order-preserving in *forward* time
    - CDS: Order-preserving in *backward* time
  - ▲ 2-D Global dynamics :
    - MDS/CDS: Both simple Global dynamics.
  - ▲ 3-D Global dynamics :
    - MDS/CDS: Both Poincaré-Bendixson Type Thm.
    - MDS: $\nexists$ stable Periodic orbit.
    - CDS: $\exists$ stable Periodic orbit.

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▲ New perspective: Cones of rank-$k$ ($k$-cone):

**Definition (Cone of rank-$k$)**

Let $k \geq 1$ be an integer. A closed subset $C \subset X$ which satisfies:

(i) $\forall x \in C, \lambda \in X$, one has $\lambda x \in C$;

(ii) $\exists$ a $k$-dim subspace $E \subset X$ s.t. $E \subset C$;

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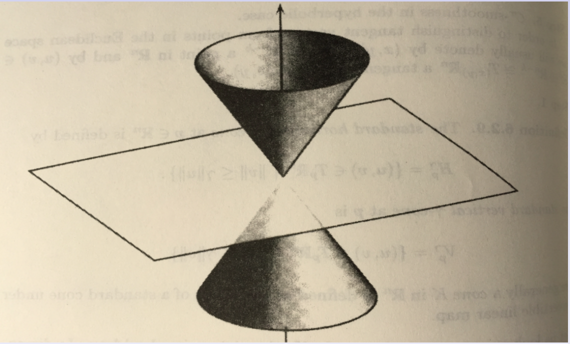
- Dynamics of Monotone semiflows w.r.t. $k$-cone.
  —— (Feng & W. & Wu [16])
- Krein-Rutmann & Perron-Frobenius Thm w.r.t. $k$-cone.
  —— (Z. Lian and W. [JDE15],[16])

# Differences between 1-cone & and $k$-cone ($k \geq 2$)



1-cone Vs. 2-cone :

- 1-cone: $C = K \cup (-K)$; (with $K \cap (-K) = \{0\}$).
- 2-cone: If write $C = \tilde{K} \cup (-\tilde{K})$, ($\tilde{K} \cap (-\tilde{K}) \neq \{0\}$).

- convexity

  —— $K$ convex. But, NO convexity for 2-cone!

# Singular perturbed MDS & CDS

- Fast-slow ODEs system:

$$\frac{dx}{dt} = f_0(x, y, \epsilon), \quad \epsilon\frac{dy}{dt} = g_0(x, y, \epsilon). \tag{$*_\epsilon$}$$

# Singular perturbed MDS & CDS

- Fast-slow ODEs system:

$$\frac{dx}{dt} = f_0(x, y, \epsilon), \quad \epsilon\frac{dy}{dt} = g_0(x, y, \epsilon). \qquad (*_\epsilon)$$

- $f_0\,(g_0) : (x, y, \epsilon) \in U \times V \times [0, \epsilon_0] \to \mathbb{R}^n\,(\mathbb{R}^m)$,
  $(U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ open bdd, $f_0, g_0 \in C_b^r$ with $r > 1)$.
- $\exists$ $C_b^r$-function $h : U \to V$ s.t. $g_0(x, h(x), 0) = 0, \forall x \in U$.
- $\mathbf{Re}[D_y g_0(x, h(x), 0)] < 0,\ \forall x \in U$.
- $\exists$ convex compact sets $D_\epsilon \subset U \times V$, conti. depend on $\epsilon$, s.t. Eq.(*) positively invariant on $D_\epsilon$.
- The limiting system

$$\frac{dx}{dt} = f_0(x, h(x), 0) \qquad (*_0)$$

is of *MDS* or *CDS* on $\mathbf{Proj_x}\,(D_0 \cap \text{graph}(h))$.

# Singular perturbed MDS & CDS

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**Theorem** (Niu & W. [16])

*Assume that*

$$\frac{dx}{dt} = f_0(x, h(x), 0) \qquad (*_0)$$

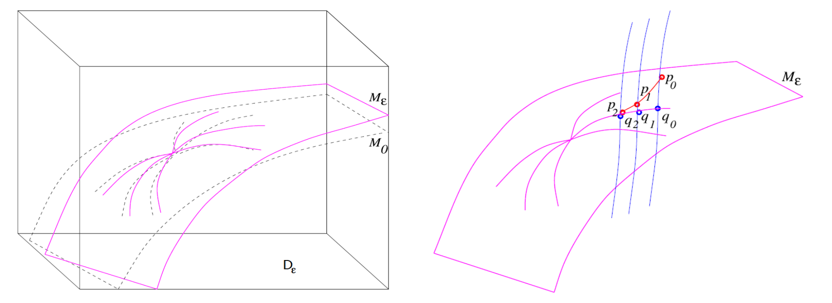
*is of* <u>*CDS*</u> *on* $\mathbf{Proj_x}\,(D_0 \cap \mathrm{graph}(h)) \subset \mathbb{R}^n$. *Then* $\exists\ \epsilon^* \in (0, \epsilon_0)$ *s.t. for each* $\epsilon \in (0, \epsilon^*)$, *we have*

(i) *If* $n = 2$, *then any orbit of* $(*_\epsilon)$ *starting from* $D_\epsilon$ *converges to some equilibrium;*

(ii) *If* $n = 3$, *then any limit-set of* $(*_\epsilon)$ *that contains no equilibrium points is a periodic orbit;*

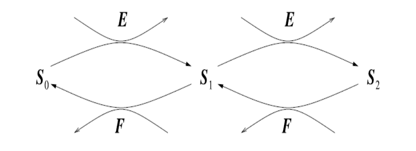
(iii) *For general* $n$, *any limit-set of* $(*_\epsilon)$ *can be embedded into an* $(n-1)$*-dim invariant hypersurface.*

▲ Singular perturbed CDS



# Some examples

▲ Dual phosphorylation Futile Cycle:




$$S_0 + E \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} C_1 \xrightarrow{k_2} S_1 + E \underset{k_{-3}}{\overset{k_3}{\rightleftarrows}} C_2 \xrightarrow{k_4} S_2 + E,$$

$$S_2 + F \underset{h_{-1}}{\overset{h_1}{\rightleftarrows}} C_3 \xrightarrow{h_2} S_1 + F \underset{h_{-3}}{\overset{h_3}{\rightleftarrows}} C_4 \xrightarrow{h_4} S_0 + F.$$

- $S_0, S_2$ : substrates; $S_1$ : Intermediate substrates.
- $E$: An Enzyme (phosphorylase-磷酸化酶): kinase-phosphorylates the substrates.
- $F$: An Enzyme (phosphatase-去磷酸酶): dephosphorylation.
- $C_i$: intermediate complex-中间化合物.

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$$S_2 + F \underset{h_{-1}}{\overset{h_1}{\rightleftarrows}} C_3 \overset{h_2}{\rightarrow} S_1 + F \underset{h_{-3}}{\overset{h_3}{\rightleftarrows}} C_4 \overset{h_4}{\rightarrow} S_0 + F.$$

$$\begin{cases} \dfrac{ds_0}{d\tau} = h_4 c_4 - k_1 s_0 e + k_{-1} c_1, \\ \dfrac{ds_2}{d\tau} = k_4 c_2 - h_1 s_2 f + h_{-1} c_3, \\ \dfrac{dc_1}{d\tau} = k_1 s_0 e - (k_{-1} + k_2) c_1, \\ \dfrac{dc_2}{d\tau} = k_3 s_1 e - (k_{-3} + k_4) c_2, \\ \dfrac{dc_4}{d\tau} = h_3 s_1 f - (h_{-3} + h_4) c_4, \\ \dfrac{dc_3}{d\tau} = h_1 s_2 f - (h_{-1} + h_2) c_3, \end{cases} \tag{DPFC}$$

# Dual phosphorylation Futile Cycle

- **Conservation Law:**

$$S = \sum_{i=0}^{2} s_i + \sum_{j=1}^{4} c_j, \ E = e + c_1 + c_2, \ F = f + c_4 + c_3.$$

- **Rescaling:**

$$x_1 = \frac{s_0}{S}, x_2 = \frac{s_2}{S}, y_1 = \frac{c_1}{E}, y_2 = \frac{c_2}{E}, y_3 = \frac{c_4}{F}, y_4 = \frac{c_3}{F}, \epsilon = \frac{E}{S}, c = \frac{F}{E}, t = \tau\epsilon.$$

- **New Equation:**

$$\begin{cases} \dfrac{dx_1}{dt} = -k_1 S x_1(1 - y_1 - y_2) + k_{-1} y_1 + h_4 c y_3, \\ \dfrac{dx_2}{dt} = -h_1 S c x_2(1 - y_3 - y_4) + h_{-1} c y_4 + k_4 y_2, \\ \epsilon \dfrac{dy_1}{dt} = k_1 S x_1(1 - y_1 - y_2) - (k_{-1} + k_2) y_1, \\ \epsilon \dfrac{dy_2}{dt} = k_3 S(1 - x_1 - x_2 - \epsilon y_1 - \epsilon y_2 - \epsilon c y_3 - \epsilon c y_4)(1 - y_1 - y_2) - (k_{-3} + k_4) y_2, \\ \epsilon \dfrac{dy_3}{dt} = h_3 S(1 - x_1 - x_2 - \epsilon y_1 - \epsilon y_2 - \epsilon c y_3 - \epsilon c y_4)(1 - y_3 - y_4) - (h_{-3} + h_4) y_3, \\ \epsilon \dfrac{dy_4}{dt} = h_1 S x_2(1 - y_3 - y_4) - (h_{-1} + h_2) y_4. \end{cases} \tag{DPFC-1}$$

# Dual phosphorylation Futile Cycle

$$D_\epsilon = \{(x_1, x_2, y_1, y_2, y_3, y_4) \in \mathbb{R}^6_+ : 0 \le y_1 + y_2 \le 1, 0 \le y_3 + y_4 \le 1,$$
$$0 \le x_1 + x_2 + \epsilon(y_1 + y_2 + cy_3 + cy_4) \le 1\}.$$

▲ $(*_0)$ turns out to be:

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{k_2 x_1}{\frac{K_1}{S} + \frac{K_1(1-x_1-x_2)}{K_2} + x_1} + \dfrac{h_4 c \frac{K_3(1-x_1-x_2)}{K_4}}{\frac{K_3}{S} + \frac{K_3(1-x_1-x_2)}{K_4} + x_2} \triangleq F_1(x_1, x_2), \\ \dfrac{dx_2}{dt} = -\dfrac{h_2 c x_2}{\frac{K_3}{S} + \frac{K_3(1-x_1-x_2)}{K_4} + x_2} + \dfrac{k_4 \frac{K_1(1-x_1-x_2)}{K_2}}{\frac{K_1}{S} + \frac{K_1(1-x_1-x_2)}{K_2} + x_1} \triangleq F_2(x_1, x_2), \end{cases}$$

where $K_1 = \frac{k_{-1}+k_2}{k_1}, K_2 = \frac{k_{-3}+k_4}{k_3}, K_3 = \frac{h_{-1}+h_2}{h_1}, K_4 = \frac{h_{-3}+h_4}{h_3}$.

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▲ $\frac{\partial F_1}{\partial x_2} < 0$, $\frac{\partial F_2}{\partial x_1} < 0$ on $\{(x_1, x_2) \mid x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 1\}$.

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$\longrightarrow$ $(*_0)$ is a *2-D CDS*.

**Corollary (Niu & W. [16])**

*$\exists$ $\epsilon^* \in (0, \epsilon_0)$ s.t. for each $\epsilon \in (0, \epsilon^*)$, any orbit of* (DPFC-1) *starting from $D_\epsilon$ converges to some equilibrium.*

Thank You for your attention!