Slow Divergence Integral and Its Application to Classical Liénard Equations of Degree 5.

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- Basic tool: the slow-divergence integral formula

  Based on works by F. Dumortier, R. Roussarie and P. Masschalck
- Main steps to prove the results

Background: Lins-de Melo-Pugh's conjecture

Consider a classical polynomial Liénard differential equation

$$\dot{x} = y - F(x),$$

$$\dot{y} = -x,$$

where $F(x)$ is a polynomial in x of degree n.

In 1977 A. Lins, W. de Melo and C. C. Pugh conjectured that the equation has

at most $\left[\frac{n-1}{2}\right]$ limit cycles,

where $\left[\frac{n-1}{2}\right]$ means the largest integer less than or equal to $\frac{n-1}{2}$.

Lins-de Melo-Pugh's conjecture

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n	3	4	5	6	7	8	$\cdots$
$\left[\frac{n-1}{2}\right]$	1	1	2	2	3	3	$\cdots$

About This Conjecture

The Lins-de Melo-Pugh's conjecture

- is true for n=3.

  - In the same paper by A. Lins, W. de Melo and C. C. Pugh:

    Lecture Notes in Math, 597（1977）.

  （can be proved by Zhang Zhifen's Theorem in a very simple way.）

- was open for $n \geq 4$ for 30 years.

- was studied by S. Smale as a "failed attempt".

    Physica D, 51（1991）; 数学译林，12（1993）.

About This Conjecture

The Lins-de Melo-Pugh's conjecture is

- not true for $n = 7$ (or $n > 7$ odd).

  - F. Dumortier, D. Panazzolo and R. Roussarie , Proc. AMS, 135（2007）.

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  - P. De Maesschalck and F. Dumortier, JDE, 250（2011）.

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- not true for $n \geq 6$: at least 2 more limit cycles can appear.

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Remarks:

1. For $n \geq 6$: can have $n - 2$ limit cycles.

   - P. De Maesschalck and R. Huzak, JDDE, 27（2015）.

2. The above 3 results were obtained by using singular perturbations.

About This Conjecture

The Lins-de Melo-Pugh's conjecture is

- true for n=4.

  - C. Li and J. Llibre, JDE, 252（2012）.

About This Conjecture

The Lins-de Melo-Pugh's conjecture is

- true for n=4.

  - C. Li and J. Llibre, JDE, 252（2012）.

- still open for $n = 5$.

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  This is the reason for us to study the classical Liénard equations of degree 5, but under singular perturbations.

Results

Consider classical Liénard equations of degree 5 under singular perturbations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) - y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon(x - \lambda(\varepsilon)),$$

where F is a polynomial of degree 5.

We denote any non-degenerate slow-fast cycle of this system by Γ_s with level s, and the slow divergence integral along Γ_s by $I(s)$.

Theorem 1 For any such Γ_s, $I(s)$ has at most one zero, and if $I(\bar{s}) = 0$ then $I'(\bar{s}) \neq 0$.

Theorem 2 The cyclicity of $\Gamma_s \leq 2$.

This means that at most 2 limit cycles of the system can be perturbed from Γ_s for small ε (including the multiplicity).

Definitions

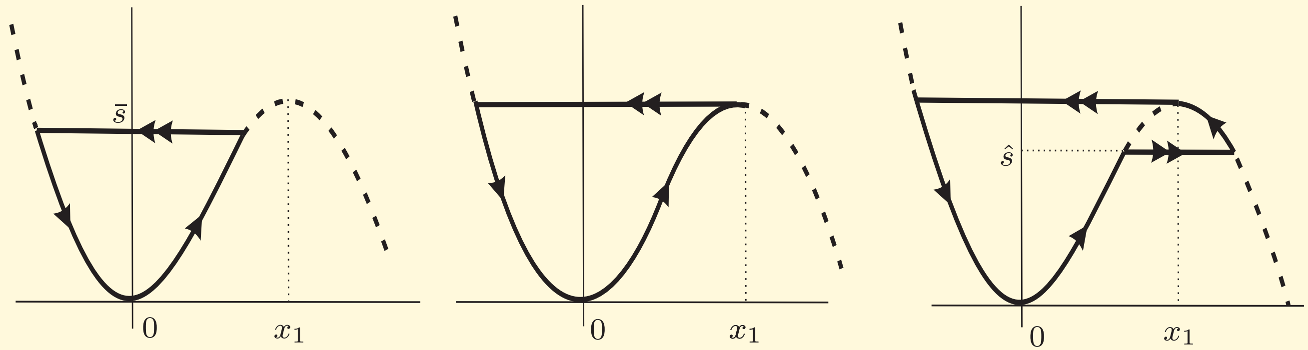
The slow curve for this system is $S_F := \{(x, y) \,|\, y = F(x)\}$.

A slow-fast cycle is formed by one or several compact parts of slow curve and one or several compact parts of fast orbits, which is homeomorphic to a circle and piecewise smooth, with uniform orientation (clockwise or counter-clockwise) coming from the fast and slow subsystems.

A slow-fast cycle Γ is non-degenerate if

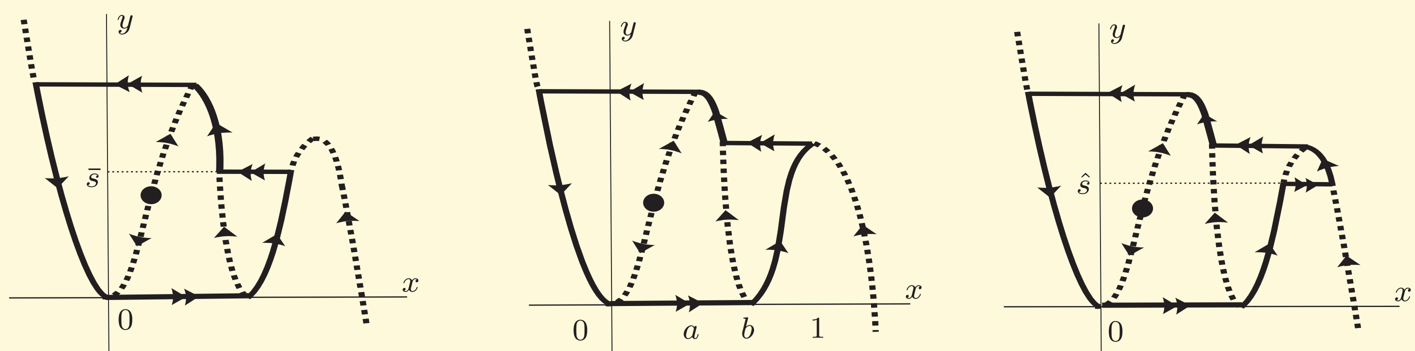
(1) for any point $(x, y) \in \Gamma \cap S_F$ if $F'(x) = 0$ then $F''(x) \neq 0$.

(2) Γ is not case II transitory, see Figs 2.



(a) cycle without head (b) case I transitory (c) cycle with head

Figure 1. Type I transitory slow-fast cycle and nearby slow-fast cycles.



(a) cycle without head (b) case II transitory (c) cycle with head

Figure 2. Type II transitory slow-fast cycle and nearby slow-fast cycles.

About Transitory Cases

Both cases I and II can appear in classical Liénard equations of degree 5.

P. De Maesschalck, F. Dumortier and R. Roussarie proved the following result:

Theorem A When the slow divergence integral is not zero for the transitory slow-fast cycle Γ of case I or II, there is at most one periodic orbit Hausdorff close to Γ for $\varepsilon > 0$ small enough. When the slow divergence integral is equal to zero, there are at most two periodic orbits Hausdorff close to Γ in case I and at most three in case II.

See: C. R. Math. Acad. Sci. Paris 352(4)(2014).

The Basic Tool: Slow Divergence Integral

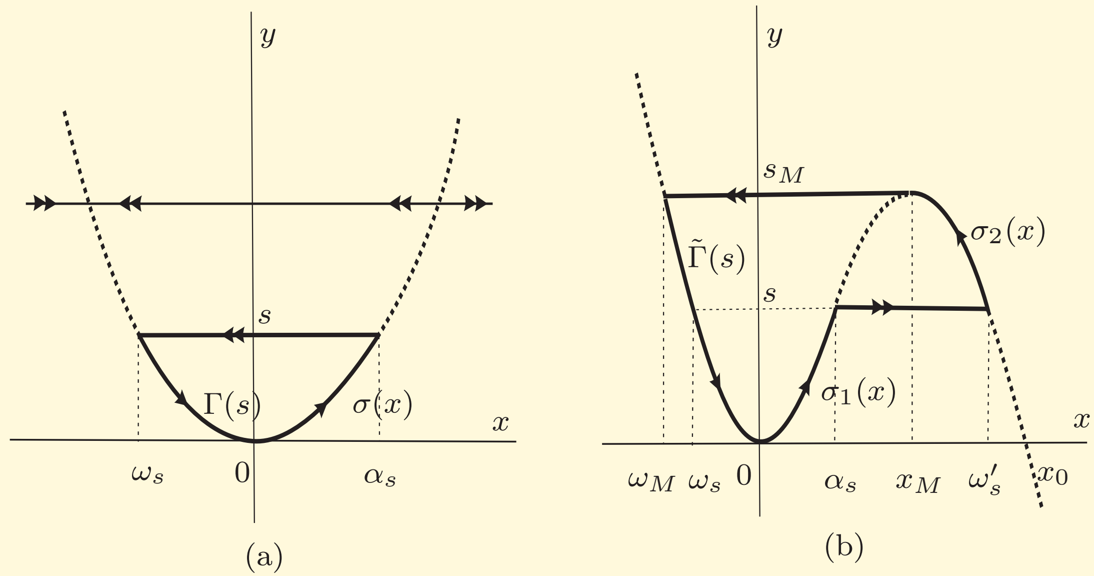


Figure 3. The slow-fast cycle $\Gamma(s)$ or $\tilde{\Gamma}(s)$.

$$\text{For } \Gamma(s): \; I(s) = \int_{\omega_s}^{\alpha_s} \frac{(F'(x))^2}{x-\lambda(0)} dx;$$

$$\text{For } \tilde{\Gamma}(s): \; \tilde{I}(s) = \int_{\omega_M}^{\alpha_s} \frac{(F'(x))^2}{x-\lambda(0)} dx + \int_{\omega'_s}^{x_M} \frac{(F'(x))^2}{x-\lambda(0)} dx.$$

New Form of The Slow Divergence Integral

If the slow curve is U-shaped, for each $x \in [\omega_s, 0]$ we define $\sigma(x) \in [0, \alpha_s]$ by

$$F(x) = F(\sigma(x)),$$

see Figure 3(a). Hence for $x \in [\omega_s, 0)$ we have that

$$\sigma'(x) = \frac{F'(x)}{F'(\sigma(x))} < 0.$$

Similarly, if the slow curve is S-shaped (see Figure 3(b)), for each $x \in [\omega_M, 0]$ we define $\sigma_1(x) \in [0, x_M]$ and $\sigma_2(x) \in [x_M, x_0]$ by

$$F(x) = F(\sigma_j(x)),\ \ j = 1, 2,$$

and for $x \neq \omega_M$, $x \neq 0$ we have that

$$\sigma_1'(x) = \frac{F'(x)}{F'(\sigma_1(x))} < 0,\ \ \sigma_2'(x) = \frac{F'(x)}{F'(\sigma_2(x))} > 0.$$

The New Form of Slow Divergence Integral

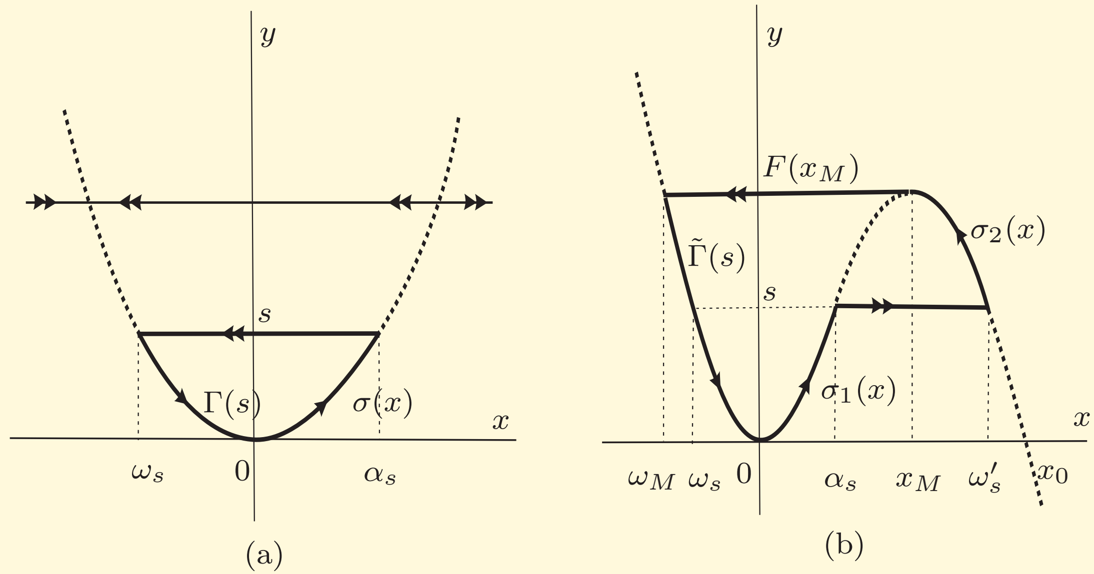


Figure 3. The slow-fast cycle $\Gamma(s)$ or $\tilde{\Gamma}(s)$.

Let $h(x) = \frac{F'(x)}{x-\lambda(0)}$,
and $x = F^{-1}(y)$ be inverse function of $y = F(x)$ for $x < 0$, then

$$I(s) = \int_0^s [h(\sigma(x)) - h(x)]|_{x=F^{-1}(y)} dy, ;$$

$$\tilde{I}(s) = \int_0^s [h(\sigma_1(x)) - h(x)]|_{x=F^{-1}(y)} dy + \int_s^{F(x_M)} [h(\sigma_2(x)) - h(x)]|_{x=F^{-1}(y)} dy.$$

The Benefits of the New formula

- In new formula the integrand function is $\frac{F'(x)}{x-\lambda(0)}$ instead of $\frac{(F'(x))^2}{x-\lambda(0)}$ in the usual formula;
- In new formula $h(\sigma(x)) - h(x) = (\sigma(x) - x)\xi(x, \sigma(x))$, where $\sigma(x) - x > 0$;
- $\xi(x, \sigma(x))$ is symmetry with respect to x and $\sigma(x)$, where $F(x) = F(\sigma(x))$. These relations may simplify the expression of $\xi(x, \sigma(x))$.

In the rest part we will introduce the main steps to prove Theorem 1, see

C. Li and K. Lu: JDE, 257 (2014), 4437–4469

for details.

Step 1: putting equation to a simpler form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) - y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon(x - \lambda(\varepsilon)),$$

where F is a polynomial of degree 5, S_F has at least one local minimum point and at least one local maximum point.

By changes of variables and parameters and using the non-degenerate condition we can suppose that S_F has a simple minimum at $(0, 0)$ and a simple maximum at $(1, 0)$; the functions $F'(x)$ and $F(x)$ can be expressed in the forms

$$F'(x) = -x(x^2 - \alpha x + \beta)(x - 1),$$

and

$$F(x) = \frac{\beta}{2}x^2 - \frac{\alpha + \beta}{3}x^3 + \frac{1 + \alpha}{4}x^4 - \frac{1}{5}x^5.$$

where $\alpha^2 \neq 4\beta > 0$.

Step 1: putting equation to a simpler form

(1) If $\alpha^2 < 4\beta$, then the minimum and maximum are unique;

(2) If $\alpha^2 > 4\beta > 0$, then without loss of generality we can suppose that S_F has two simple minimum points at $(0,0)$ and $(b,0)$, and has two simple maximum points at $(a,0)$ and $(1,0)$, where

$$0 < a < b < 1.$$

In this case $F'(x)$ and $F(x)$ has the forms

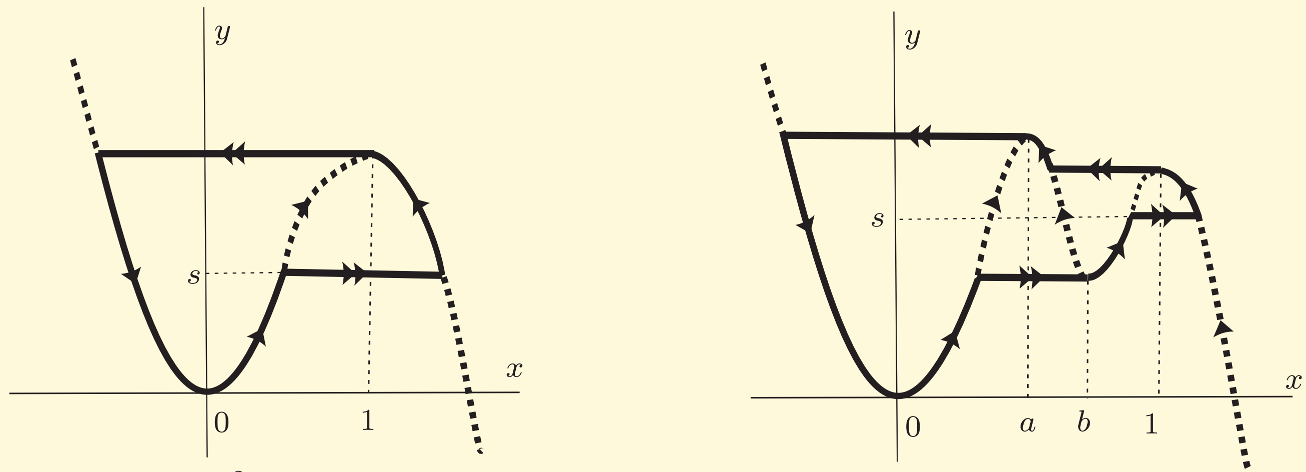
$$F'(x) = -x(x-a)(x-b)(x-1),$$

and

$$F(x) = \frac{ab}{2}x^2 - \frac{a+b+ab}{3}x^3 + \frac{1+a+b}{4}x^4 - \frac{1}{5}x^5,$$

where $\alpha = a+b$ and $\beta = ab$.

Examples of slow-fast cycles and corresponding $I(h)$



(a) $\alpha^2 < 4\beta$ (b) $\alpha^2 > 4\beta$

Fig 4. Different shapes of slow-fast cycles.

$$(a)\ I(s) = \int_0^s [h(\sigma_1(x)) - h(x)]|_{x=F^{-1}(y)} dy + \int_s^{F(1)} [h(\sigma_2(x)) - h(x)]|_{x=F^{-1}(y)} dy.$$

$$(b)\ I(s) = \int_0^{F(b)} [h(\sigma_1(x)) - h(x)]|_{x=F^{-1}(y)} dy + \int_{F(b)}^{s} [h(\sigma_3(x)) - h(x)]|_{x=F^{-1}(y)} dy$$
$$+ \int_s^{F(1)} [h(\sigma_4(x)) - h(x)]|_{x=F^{-1}(y)} dy + \int_{F(1)}^{F(a)} [h(\sigma_2(x)) - h(x)]|_{x=F^{-1}(y)} dy.$$

A basic Lemma

Note that

$$h(\sigma_k(x)) - h(x) = (\sigma_k(x) - x)\,\xi(\sigma_k(x), x).$$

Lemma For classical Liénard equations of degree 5 we have that if for $x < 0$ each function $\xi(\sigma_k(x), x)$ has at most one zero for $k = 1, 2, \cdots, \ell$ ($\ell = 2$ or 4), then the slow divergence integral of any slow-fast cycle Γ of the system has at most 1 zero and the zero is simple when exists.

Remark To prove Theorem 1, we only need to prove that

for $x < 0$ each function $\xi(\sigma_k(x), x)$ has at most one zero for $k = 1, 2, \cdots, \ell$ ($\ell = 2$ or 4).

Step 2: the position of canard point

We will consider 3 cases:

(1) the canard point is at $(0, 0)$;

(2) the canard point is at $(a, F(a))$;

(3) there is no canard point (all of turning points are jump points).

Remark : If the canard point is at $(1, F(1))$ or $(b, F(b))$, then by the change $(x, y, \lambda) = (1 - \bar{x}, F(1) - \bar{y}, 1 - \bar{\lambda})$, the system keeps the same form, but $F(x)$ is replaced by $\bar{F}(\bar{x}) = F(1) - F(1 - \bar{x})$, and the parameters (a, b) with $0 < a < b < 1$ become $(\bar{a}, \bar{b}) = (1 - b, 1 - a)$ with $0 < \bar{a} < \bar{b} < 1$. Moreover, along the slow curve the original maximal point $(1, F(1))$ becomes a minimal point $(0, 0)$, and the original minimal point $(b, F(b))$ becomes a maximal point $(\bar{a}, \bar{F}(\bar{a})) = (1 - b, F(1) - F(b))$.

Step 3: the canard point is at $(0,0)$

$$I(s) = \sum_{j=1}^{\ell} \int_{s_{j-1}}^{s_j} [(h(\sigma_j(x)) - h(x)]|_{x=F^{-1}(y)} dy,$$

$$= \sum_{j=1}^{\ell} \int_{s_{j-1}}^{s_j} [(\sigma_j(x) - x)\,\xi(\sigma_j(x), x)]|_{x=F^{-1}(y)} dy,$$

where $\sigma_j(x) - x > 0$, and

$$h(x) = \frac{F'(x)}{x - \lambda(0)} = \frac{F'(x)}{x} = -(x^2 - \alpha x + \beta)(x-1).$$

Hence

$$\xi(\sigma_j(x), x) = -(x^2 + \sigma_j^2(x)) - x\sigma_j(x) + (\alpha+1)(x + \sigma_j(x)) - (\alpha+\beta)$$

$$= -\left(x + \bar{x} - \frac{1+\alpha}{2}\right)^2 - \left(\beta - \frac{(1-\alpha)^2}{4}\right) + x\bar{x},$$

where $\bar{x} = \sigma_j(x)$, hence $x\bar{x} < 0$, and if $\beta - \frac{(1-\alpha)^2}{4} \geq 0$, we have $\xi(\bar{x}, x) < 0$, hence $I(h)$ has a fixed sign and the proof is complete.

So we suppose

$$\beta < \frac{(1-\alpha)^2}{4},$$

and prove that $\xi(\bar{x}, x) = 0$ has at most one zero for $x < 0$.

Thus we only need to consider

$$(\alpha, \beta) \in \{\Omega_1 \cup \Omega_2\},$$

where

$$\Omega_1 = \{(\alpha, \beta) \,|\, \alpha^2/4 < \beta < (1-\alpha)^2/4, -\infty < \alpha \leq \frac{1}{2}\},$$
$$\Omega_2 = \{(\alpha, \beta) \,|\, 0 < \beta < \min\,[\alpha^2/4, (1-\alpha)^2/4], 0 < \alpha < 1\}.$$

Note that $\Omega_1 \cup \Omega_2$ is divided in the regions a, b, c, d, e and f by curves C_1-C_4 and lines L_1 and L_2, see Fig 5.

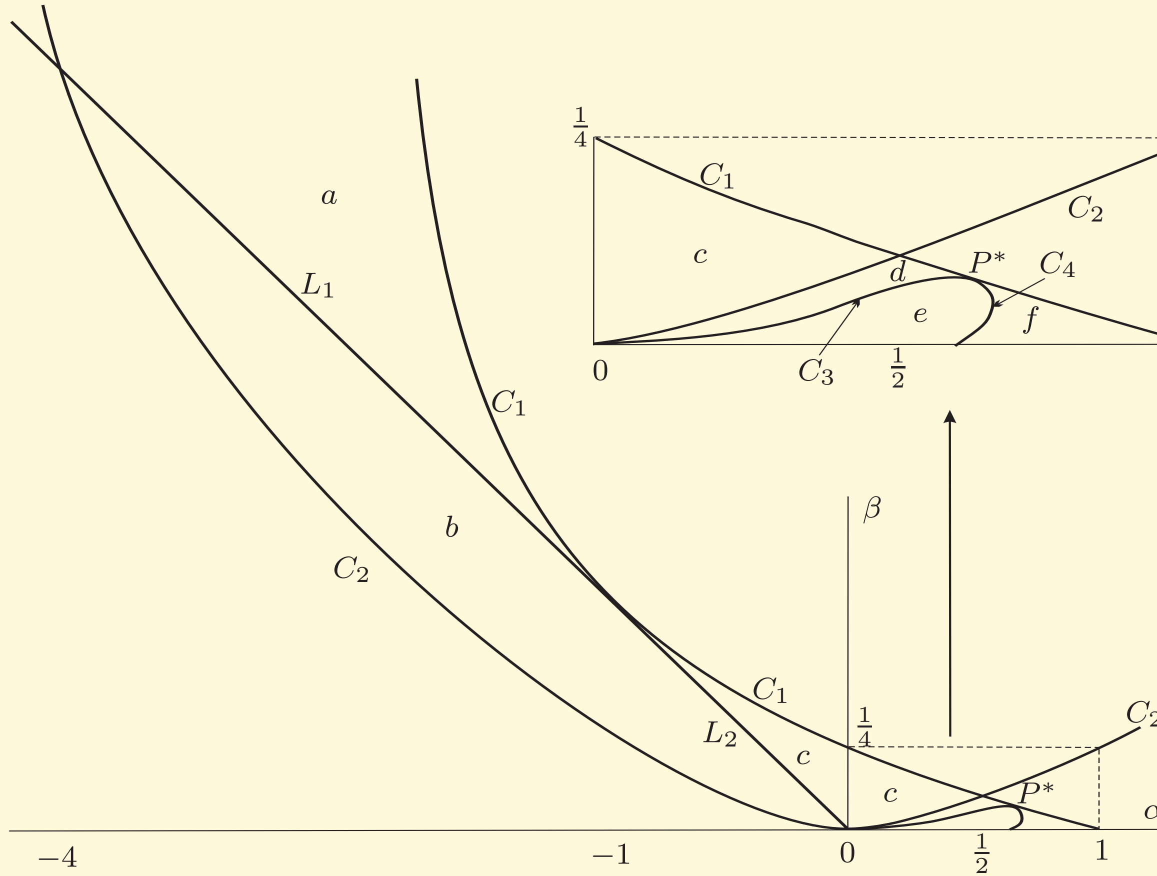


Fig 5. Partitions of Ω_1 and Ω_2.

Conclusion

$$\sharp\{\xi(\bar{x}(x),x)=0\,|\,x<0\}=0\quad \text{if } (\alpha,\beta)\in a,c,d,f,L_1,L_2,C_3,C_4;$$

$$\sharp\{\xi(\bar{x}(x),x)=0\,|\,x<0\}=1\quad \text{if } (\alpha,\beta)\in b,e;$$

Method to prove

$$\xi(\bar{x},x)=0,\qquad \eta(\bar{x},x)=0,$$

the second comes from $F(x)=F(\bar{x})$. Eliminating $\bar{x}$, we obtaion

$$\psi(x)=0,$$

where

$$
\begin{aligned}
\psi(x) = {} & 144x^8 - 324(1+\alpha)x^7 + 27(7+30\alpha+16\beta+7\alpha^2)x^6 \\
& -6(9+91\alpha+184\beta+91\alpha^2+64\alpha\beta+9\alpha^3)x^5 + 3(18+39\alpha \\
& +237\beta+154\alpha^2+428\alpha\beta+112\beta^2+39\alpha^3-33\alpha^2\beta+18\alpha^4)x^4 \\
& -36(\alpha+\beta)(3+2\alpha+33\beta+2\alpha^2-7\alpha\beta+3\alpha^3)x^3 \\
& +(30\alpha+120\beta+\alpha^2+302\alpha\beta+931\beta^2-10\alpha^3+268\alpha^2\beta \\
& +56\alpha\beta^2+48\beta^3+\alpha^4-28\alpha^3\beta-119\alpha^2\beta^2+30\alpha^5+30\alpha^4\beta)x^2 \\
& -2(\alpha+\beta)^2(15-19\alpha+176\beta-19\alpha^2-64\alpha\beta+15\alpha^3)x \\
& +(\alpha+\beta)\zeta(\alpha,\beta) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta(\alpha,\beta) = {} & 64\beta^3 - (15\alpha^2+258\alpha-465)\beta^2 + (60\alpha^3-18\alpha^2-180\alpha+90)\beta \\
& -\alpha^2(5\alpha-3)(3\alpha-5).
\end{aligned}
$$

We have that

- $\psi(-\infty) = +\infty$, $\psi(0) > 0$ if $(\alpha, \beta) \in a, c, d, f$; $\psi(0) < 0$ if $(\alpha, \beta) \in b, e$.
- $\psi(0) = 0$ if $(\alpha, \beta) \in L_1, L_2, C_3, C_4$ (on the boundaries of the above sub-regions).
- For $x < 0$ and $(\alpha, \beta) \in [\Omega_1 \cup \Omega_2]$, if $\psi(x) = 0$ than $\psi'(x) \neq 0$.
- By using the above information and the variation argument.
- The behavior of $\psi(x)$ for $(\alpha, \beta) \in C_2$ is shown in Fig 6.

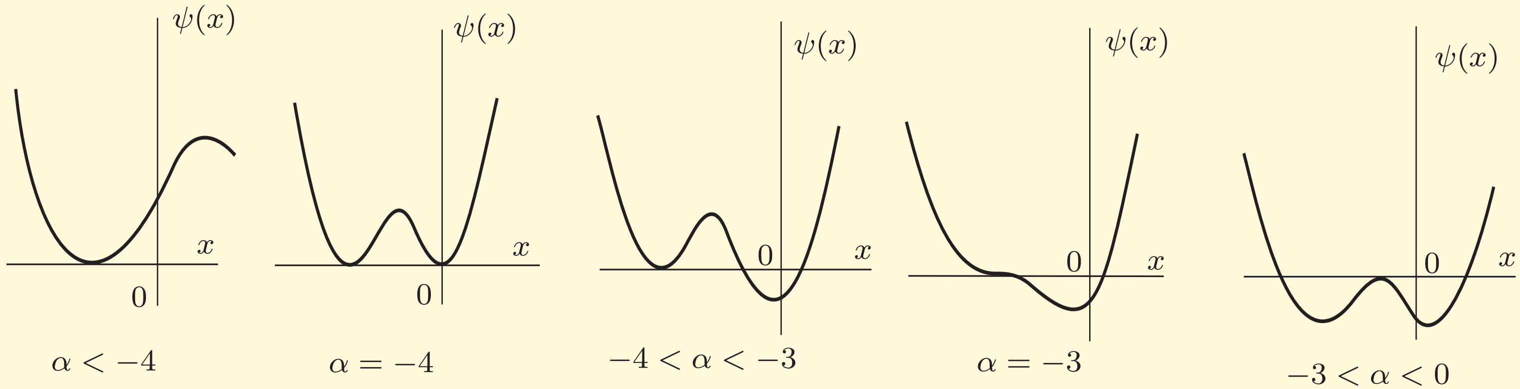


Fig 6. The behavior of $\psi(x)$ for $x \leq 0$, $(\alpha, \beta) \in C_2$ and $\alpha < 0$.

Step 4: the canard point is at $(a, F(a))$

In this case C_F has two minimum at $(0,0)$ and $(b, F(b))$, two maximum at $(a, F(a))$ and $(1, F(1))$. We classify C_F to following 11 cases, corresponding to 11 subregions in $\mathcal{D} = \{(a,b) \,|\, 0 < a < b < 1\}$, see Fig 7.

(1) $F(b) > 0, F(1) > F(a)$;

(2) $F(b) > 0, F(1) = F(a)$;

(3) $F(b) > 0, F(1) < F(a)$;

(4) $F(b) = 0, F(1) > F(a)$;

(5) $F(b) = 0, F(1) = F(a)$;

(6) $F(b) = 0, F(1) < F(a)$;

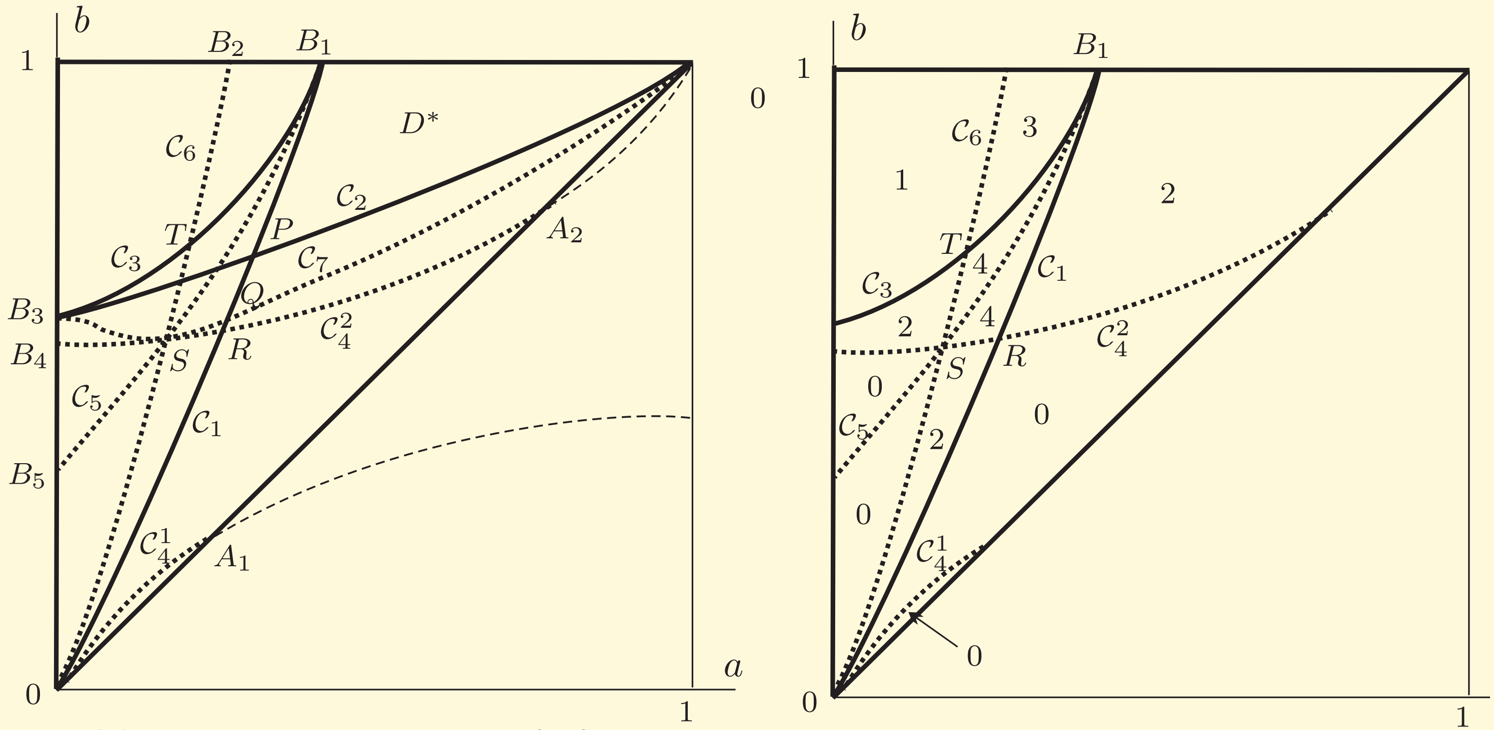
(7) $F(b) < 0, F(1) > F(a)$;

(8) $F(b) < 0, F(1) = F(a)$;

(9) $F(b) < 0, 0 < F(1) < F(a)$;

(10) $F(b) < 0, F(1) = 0$;

(11) $F(b) < 0, F(1) < 0$.



(a) The partition of $\mathcal{D}$ by $\{\mathcal{C}_j\}$

(b) Numbers $\mathcal{N}[(0,b)]$ in open regions

Fig 7. Partition of $\mathcal{D}: 0 < a < b < 1$ and distribution of $\mathcal{N}[(0,b)]$.

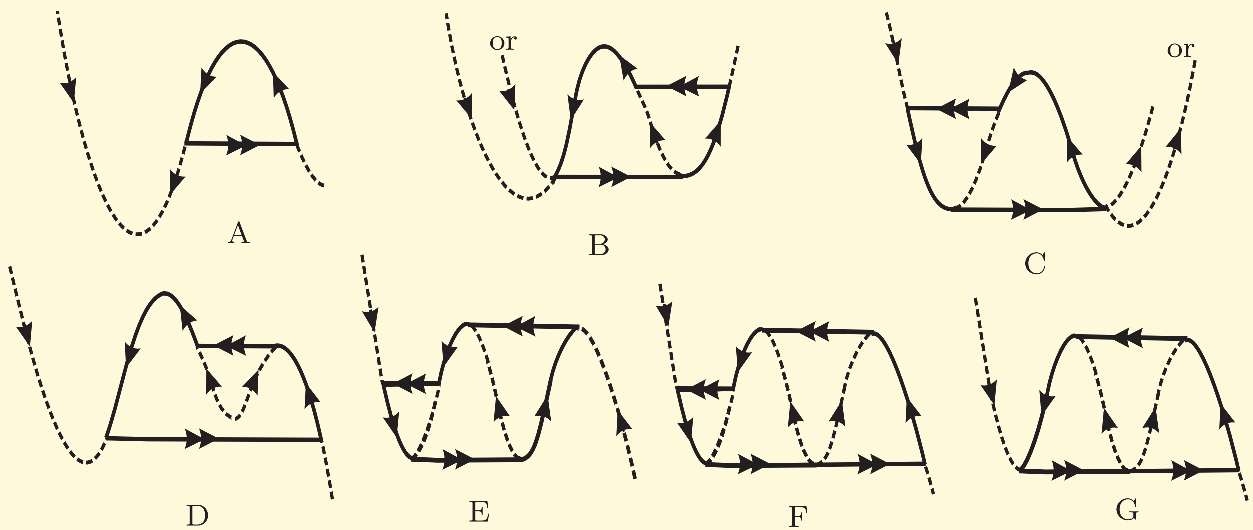


Fig 8. Some shapes of slow-fast cycles containing $(a, F(a))$ as a canard point.

Remark: we leave the two layers case for further study.

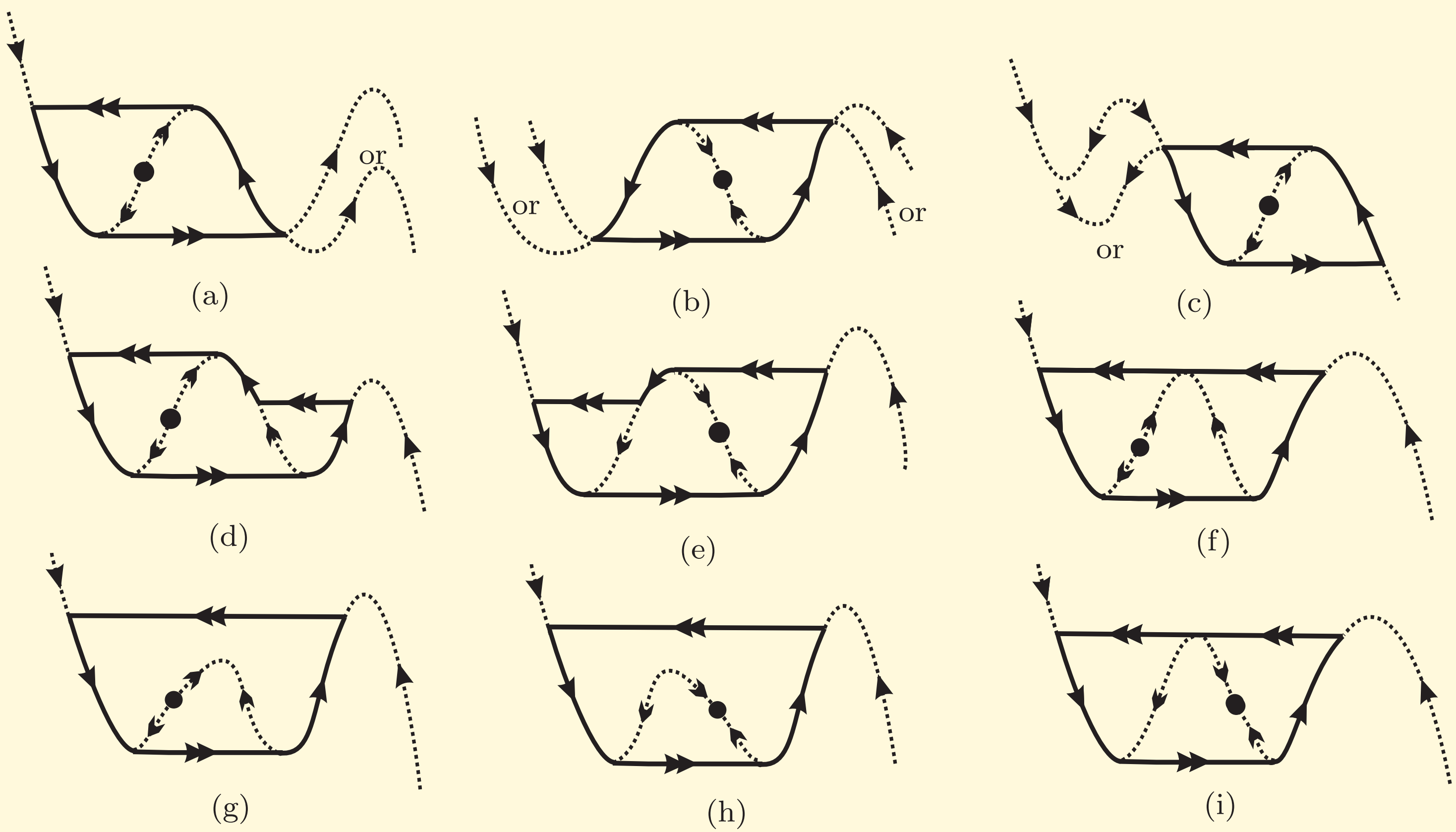


Fig 9. Some shapes of slow-fast cycles without canard point.

谢谢大家!

THANK YOU VERY MUCH!